A General Method for Facilitating the Solution of Kepler's Equation by Mechanical Means. By T. J. J. See, A.M., Ph.D. (Berlin).

The standard works on planetary motion, such as Gauss' *Theoria Motus*, Oppolzer's *Bahnbestimmung*, and Watson's *Theoretical Astronomy*, give methods for solving Kepler's equation which are very satisfactory when the eccentricity of the orbit is small, and also when this element is large, as in the case of most of the periodic comets. When the eccentricity is small, an expansion in series, usually by Lagrange's Theorem, enables us to find the eccentric anomaly with the desired facility. The series frequently employed has the form

$$E_0 = M + e'' \sin M + e''\left(\frac{e}{2}\right) \sin 2M + \ldots$$

To the approximate value E_0, obtained from a few terms of this series, we apply a correction resulting from the expansion by Taylor's Theorem :

$$E = E_0 + \frac{dE_0}{dM_0} dM_0 + \ldots$$

The equation of Kepler gives

$$\frac{dM_0}{dE_0} = 1 - e \cos E_0 ;$$

and since

$$dM_0 = M - M_0,$$

we find two terms of the series to be

$$E = E_0 + \frac{M - M_0}{1 - e \cos E_0}.$$

Successive applications of this formula will readily yield the true value of the eccentric anomaly. But when the eccentricity is considerable the expansion in series fails to converge with the desired rapidity. On the other hand, when the orbits differ but little from parabolas, the solution can be readily found by means of special tables, such as those given by Gauss, Watson, and Oppolzer.

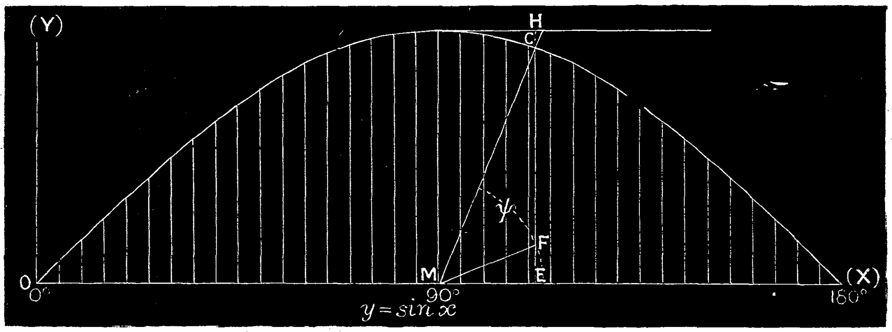
It is very remarkable that among the many solutions of Kepler's equation discovered by mathematicians there is not one, so far as I am aware, which has come into general use among astronomers that is applicable to ellipses of all possible eccentricities. The method to which I desire to call attention is a modification of the graphical method given by Klinkerfues,* and originally proposed by Dubois.†

* *Theoretische Astronomie*, p. 17.
† *Astronomische Nachrichten*, No. 1404.

Suppose we construct, on a convenient scale, a semicircumference of the curve of sines, $y = \sin x$. In practice it is desirable to use millimetre paper, and a convenient scale is obtained by taking one degree of the arc as five millimetres, so that the scale may easily be read to 0°·1. The origin of the arc is taken at the origin of coordinates; and as the scale along the axis of abscissæ extends from 0° to 180°, it will have a length of 90 centimetres.

In the figure let OM represent the mean anomaly, and suppose from M we draw a right line making an angle Ψ with the axis of abscissæ, the angle Ψ being defined by the equation

$$\tan \Psi = \frac{1}{e}.$$



Let the abscissa of the point C, determined by the intersection of the right line MC with the sine curve, be denoted by E. Then we evidently have

$$\text{OE} - \text{ME} = \text{OM}.$$

Thus, denoting the arc OE by E, and observing that $e \sin \Psi = \cos \Psi$, we find that $e \sin \Psi = \text{ME}$, the radius in case of $\sin \Psi$ being such that $\sin \Psi$ is always equal to sin E.

Hence we get

$$\text{OE} - \text{ME} = \text{OM},$$

or

$$\text{E} - e \sin \text{E} = \text{M},$$

which is the equation of Kepler.

Therefore we conclude that if for an orbit of given eccentricity we construct a triangle CME (in practice this may be made of cardboard) and apply the vertex M of the triangle to the successive mean anomalies, the base coinciding with the x axis, the intersection of the hypothenuse with the curve of sines will give at once abscissæ which are the corresponding eccentric anomalies. Any actual diagram such as we have described will be subject to slight inaccuracies of construction owing to the transcendental nature of the sines, and hence we cannot obtain solutions of absolute precision. But it is entirely possible to get approximate solutions exact to 0°·1, and this work

can be done with the greatest rapidity. It is merely necessary to slide the base of the triangle along the x axis, placing the vertex M at the points corresponding to the different values of the mean anomaly, and reading off the corresponding eccentric anomalies.

This triangle device is rendered possible by virtue of the fact that Ψ is constant in $\tan \Psi = \frac{1}{e}$; and we may observe that in case of elliptic orbits the angle Ψ can only vary from 45° in case of a parabola to 90° in case of a circle. This method is therefore directly applicable to ellipses of every possible eccentricity, and the accuracy of the solution is always substantially the same. In case of parabolic motion, however, the method fails, since when $\Psi = 45°$ the hypothenuse MC is tangent to the sine curve at the origin. But for $e < 1$ the hypothenuse MC intersects the curve $y = \sin x$, and the intersection will be well defined except when e approaches unity and M is very small. In such cases it is best to use the Special Tables or the Theory of Parabolic Motion. Solutions exact to 0°·1 are often sufficient in the present state of double-star observation, and we readily see how great is the practical value of this method in comparing a long series of observations with a given set of elements. One hundred approximate solutions of Kepler's Equation, accurate to 0°·1, may be obtained by this method in less than half an hour; while if e lies between 0·35 and 0·85 a skilled computer could probably not obtain the same results by the ordinary method in less than a day. Thus the time and labour required for this work is much diminished, and it is clear that the chances of large error are correspondingly reduced.

If a curve of sines were engraved on a metallic plate it would be an easy matter to devise a movable protractor which could be set at any angle; such a piece of apparatus would serve for every possible elliptic orbit, and would last for an indefinite time. Considering the immense labour devolving upon astronomers in the computation of the motion of the heavenly bodies, it would seem that such a labour-saving device might be advantageously employed in the offices of the astronomical ephemerides. However, as several astronomers have prepared tables for facilitating the solution of Kepler's Equation in case of orbits which are not very eccentric, such an apparatus would be useful chiefly in work on the more eccentric asteroids, the double stars, and periodic comets. In dealing with the motions of these bodies the labour saved would be very considerable, and we might hope that the apparatus here suggested would come into actual use. But in case this instrument of precision could not be successfully manufactured, owing to its limited commercial use, it is easy for a working astronomer to construct a curve of sines on millimetre paper. This can be mounted on a suitable wooden board, and a triangle of cardboard will give the solutions of Kepler's Equation for any given orbit.

Klinkerfues has shown how we may use the diagram to correct our approximate values of E with great rapidity. Suppose we take a pair of dividers and set them to the value of e on the scale of the sine curve; then, placing one foot at M, let us describe a circular arc cutting the ordinate CE in F. The line $MF=e$, $ME=e \sin E$, and $FE=e \cos E$. Hence

$$HF = 1 - e \cos E = \frac{dM}{dE}.$$

The exact value of M is given, and since

$$M_0 = E_0 - e \sin E_0 = OE - ME,$$

the value of $dM_0 = M - M_0$ is at once obtained. Then we have

$$\Delta E_0 = \frac{M - M_0}{1 - e \cos E_0},$$

which gives a second approximation to the true value of the eccentric anomaly. This correction can be applied with great rapidity, since the dividers will always have the same setting for a given eccentricity.

Thus, while the graphical method, originally proposed by Dubois and afterwards improved by Klinkerfues, has been suggested many years ago, it does not appear that it has ever come into actual use; and I have therefore thought it worthy of the attention of astronomers, particularly in the modified form here proposed. For it seems clear that a general method for solving Kepler's Equation by mechanical means is an urgent *desideratum* of astronomy, owing to the large number of orbits requiring investigation and the wide range of their eccentricities.

In computing the motions of double stars, where the eccentricities have all values from 0·1 to 0·9, I have found this device of the greatest service. It will be almost, if not quite, equally important in case of the periodic comets and the asteroids. But in dealing with comets and planets, where we desire very exact solutions of Kepler's Equation, I would suggest that the approximate values of E should be read off, and one correction applied in the manner here indicated; a second correction to be *computed* by the formula

$$\Delta E_0' = \frac{M - M_0'}{1 - e \cos E_0'},$$

where M_0', E_0' are the first corrected values of E_0, or

$$E_0 + \frac{M - M_0}{1 - e \cos E_0}$$

will probably give the true values of the eccentric anomaly

$$E = E_0' + \Delta E_0',$$

within a small fraction of a second of arc. Thus practically all *computation* is done away with, except a single correction which would be required in order to insure the accuracy desirable in planetary and cometary ephemerides.

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On a Catalogue of Stars in the Calendarium of Mohammad Al Achsasi Al Mouakket. By E. B. Knobel.

I have lately obtained a 17th century Arabic MS. of a calendarium by Al Achsasi—a name that is not to be found in the usual sources of information about Oriental astronomers, such as D'Herbelot, *Bibliothèque Orientale*; Casiri, *Bibliotheca Escurialensis*; Houzeau, *Bibliographie de l'Astronomie*; the British Museum, the Bodleian, and the Leyden Libraries, &c. But fortunately I have found the name of the author and the title of the treatise in the catalogue of the Khedive's library at Cairo, but without any date.

The author, Mohammad Al Achsasi Al Mouakket, was Sheikh of the Grand Mosque at the College at Cairo, and surnamed Al Mouakket from his functions of regulating the times and the hours at the mosque. He derived his name "Achsasi" from a village so called, which I find existed in the province of Fayoum in Egypt. The MS. was therefore clearly written at Cairo. Its title is "Durret Al Muddiah Fih Al Aamal Al Shamsiah" (Pearls of brilliance upon the solar operations).

The MS. is unfortunately not complete, but it contains tables of the position of the Sun for each day of the Coptic and Syriac months, a list of cities in Arabia and Northern Africa, a concordance of Coptic and Mohammadan years, a catalogue of stars, a complete almanac of the Coptic year, and some other astronomical tables.

The catalogue gives the positions of 112 stars in right ascension and declination; and as it presents some points of interest I have translated this portion of the MS. as perhaps worthy of publication.

Prior to modern European astronomy it was unusual to give the positions of stars in right ascension and declination, and in the few cases where this was done by Oriental astronomers the right ascensions, designated by the term al mataliā "coascendants," were always calculated from the first point of Capricornus —the solstitial colure, as we find in the present instance.

In investigating this MS. it is natural to suspect that it is a copy of the Calendarium of Al Tizini, who also bore the designation of "Al Mouakket," and who published his tables at Damascus in 1534. Al Tizini's catalogue of 302 stars was translated by